Using Complex Conjugated Magnitudes- and Orthogonal Park/Clarke Transformation Methods of DC/AC/AC Frequency Converter

B. Dobrucky, M. Benova, P. Spanik
University of Zilina, Univerzitna1, Zilina, Slovakia, phone: +421 41 5131602; e-mail: dobrucky@fel.uniza.sk

Introduction
The paper deals with mathematical modeling of two-stage frequency converter system with induction motor. There are used two special methods of investigation: method of complex conjugated amplitudes, and the orthogonal Park/Clarke transformation method. The first one is used for steady-state investigation; the second one is suitable for investigation of three-phase electric circuits. The combination of both methods is very useful for analysis of three-phase electric motors in steady-state condition: constant angular speed, and when operator \( \frac{d}{dt} \) in its dynamical model is substituted by operator \( j \nu \omega t \). Then the effect of individual harmonic components on motor properties can be investigated.

Methods Used for Modelling and Calculation
Method of complex amplitudes has been introduced by Takeuchi \[1\] for analysis of converter circuit supplied electric machines in steady-state. The principle is based on substitution of trigonometric function by exponential one with complex argument. After determination of investigated variable in complex form, the variable can be then transformed back into time domain. Regarding to nonharmonic time waveforms of converter quantities the Fourier analysis is used for variables as the first step.

Method of orthogonal transformation for electrical quantities was introduced by Park \[2\] for three-phase electric machines. The method makes it possible to transform symmetrical 3-phase system into equivalent two-phase orthogonal system. This transformation decreases number of differential equations (from 3 to 2), and removes variable coefficients in the equations. Besides, trajectories of the quantities in complex Gauss plane denote themselves by six-side symmetry, thus the steady-state quantities can be calculated in only one sixth of time period. Clarke's multiplicative transformation constant (equal 2/3) provides the invariances of voltage and current quantities in the both coordinating systems.

Using of Complex Conjugated Amplitude Methods for Electrical Circuit Fed by Single-Phase Inverter
For rectangular form of electric voltage with cosine harmonic components, the sum of its odd harmonic components can be written as:

\[
 u(t) = \frac{4U_0}{\pi} \sum_{v=0}^{\infty} \left( -1 \right)^{(2v+1)} \frac{\cos(2v+1)0t}{(2v+1)}
\]

for non-negative integer \( v \) (from interval \( 0; \infty \)), constant supply voltage of inverter \( U_0 \) and constant angular frequency \( (\omega = 2\pi f_0) \)

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Using Euler relations the non-harmonic voltage can be expressed as

\[
u(t) = \frac{2U_0}{\pi} \sum_{v=0}^{\infty} (-1)^{2v+1} e^{j(2v+1)\omega t} + e^{-j(2v+1)\omega t} \frac{1}{(2v+1)} . \tag{3}\]

Then corresponding complex current is

\[
i(t) = \frac{2U_0}{\pi} \sum_{v=0}^{\infty} (-1)^{2v+1} e^{j(2v+1)\omega t} + e^{-j(2v+1)\omega t} \frac{1}{(2v+1)(R + j(2v+1)\omega L)} . \tag{4}\]

that can be written in complex conjugated magnitude form:

\[
i(t) = \frac{1}{\sqrt{2}} \sum_{v=0}^{\infty} (l_{2v+1} e^{j(2v+1)\omega t} + l_{2v+1}^* e^{-j(2v+1)\omega t}) , \tag{5a}\]

where complex magnitude of current will be

\[
l_{2v+1} = \frac{2\sqrt{2}}{\pi} \frac{(-1)^{2v+1}}{2v+1} \frac{U_0 e^{-j\theta_{2v+1}}}{\sqrt{R^2 + (2v+1)^2 \omega^2}} \tag{5b}\]

and complex conjugate current magnitude

\[
l_{2v+1}^* = \frac{2\sqrt{2}}{\pi} \frac{(-1)^{2v+1}}{2v+1} \frac{U_0 e^{j\theta_{2v+1}}}{\sqrt{R^2 + (2v+1)^2 \omega^2}} . \tag{5c}\]

Finally, by adapting of (5) and by substitution in the equation (4) with using (2), for current form in time domain one will obtain

\[
i(t) = \frac{4U_0}{\pi} \sum_{v=0}^{\infty} \frac{(-1)^{2v+1}}{(2v+1)^2} \frac{\cos[(2v+1)\omega t - \varphi_{2v+1}]}{\sqrt{R^2 + (2v+1)^2 \omega^2}} . \tag{6}\]

**Note:** Eq. (6) is approximated numerical solution of ordinary differential equation

\[
\frac{di(t)}{dt} = \frac{R}{L} \cdot i(t) + \frac{1}{L} \cdot u(t) . \tag{7}\]

The relation for resulting current wave-form can be obtained also in compact closed form using classical analytical solution, Laplace transform or z-transform [3]

\[
i(t) = \frac{U_0}{R} \left[ 1 - e^{-t/\tau} \right] + I_0 \tag{8}\]

and

\[
i(t) = \frac{U_0}{R} \frac{1 - e^{-T/2\tau}}{1 + e^{-T/2\tau}} = \frac{U_0}{R} \tanh \left( \frac{T}{4\tau} \right) , \tag{9}\]

where \( \tau \) – time constant of the circuit, \( \tau = L/R \).

Anyway, the solution (6) makes it possible to analyse more exactly each harmonic component (of current) comprised in total waveform

\[
i_{2v+1}(t) = \frac{4U_0}{\pi} \frac{(-1)^{2v+1}}{2v+1} \frac{1}{|Z_{2v+1}|} \cos[(2v+1)\omega t - \varphi_{2v+1}] . \tag{10}\]

The waveforms whose Fourier series analysis leads to sine functions of the harmonics can be expressed by the similar way, Fig. 2.

\[
u(t) = \frac{4U_0}{\pi} \sum_{v=0}^{\infty} \frac{\sin[(2v+1)\omega t]}{(2v+1)} , \tag{11}\]

\[
i(t) = \frac{4U_0}{\pi} \sum_{v=0}^{\infty} \frac{\sin[(2v+1)\omega t - \varphi_{2v+1}]}{(2v+1)(R^2 + (2v+1)^2 \omega^2 L^2)} . \tag{12}\]

**Fig. 2.** Rectangular time-waveform of single phase inverter voltage with sine harmonic components.

The rectangular pulse time-waveforms as is shown on Fig.3 can be expressed by the similar way

\[
u(t) = \frac{4U_0}{\pi} \sum_{v=0}^{\infty} \frac{\cos[(2v+1)\omega t]/6}{(2v+1)} \sin[(2v+1)\omega t] , \tag{13}\]

\[
i(t) = \frac{4U_0}{\pi} \sum_{v=0}^{\infty} \frac{\cos[(2v+1)\omega t]/6}{(2v+1)(R^2 + (2v+1)^2 \omega^2 L^2)} \sin[(2v+1)\omega t - \varphi_{2v+1}] . \tag{14}\]

**Fig. 3.** Rectangular pulse time-waveform of single phase of three phase’s inverter voltage

**Electrical model of 2-stage converter**

Scheme of the 2-stage converter is shown in Fig. 4. It comprises two semiconductor type of converters [4]:

- single-phase voltage inverter as the first stage,
- three-phase matrix converter or cycloconverter as the second stage.

The first stage operates with constant voltage \( U_0 \) and fixed frequency \( f_0 \). The second one supplies passive R-L or active load (electric motor) with variable output frequency and which is much lesser then frequency of AC interlink between stages.
Considering rectangular form of the phase-voltage length of 2\pi/3 radians with \( I_0 \) equal \( U_0/R \), the scheme can be reconfigured to the scheme of three-phase current inverter with IM motor load [4]. Fig. 5, whereas commutating capacitors could be omitted because of switches of inverter are switch-off capability. Control of such system is described in greater detail in [5].

![Fig. 4. Overall schematic diagram of 2-stage 3-phase DC/AC/AC converter](image)

Fig. 4. Overall schematic diagram of 2-stage 3-phase DC/AC/AC converter

The coefficients for odd \( n (=2\nu+1) \) will then be

\[
a_{2\nu+1} = \frac{2U_0}{\pi} \sin \left(2\nu+1\right) \frac{2\pi}{3},
\]

(16a)

\[
b_{2\nu+1} = \frac{2U_0}{\pi} \left(1-\cos (2\nu+1) \frac{2\pi}{3}\right).
\]

(16b)

Using equations (15)-(16b) and complex magnitudes method, the voltage will be

\[
u(t) = \frac{4U_0}{\pi} \sum_{\nu=0}^{\infty} \frac{\sin \left(2\nu+1\right) \frac{\pi}{3} \cos \left(2\nu+1 \left(\omega t - \frac{\pi}{3}\right)\right)}{2\nu+1}.
\]

(17)

**Implementation of orthogonal transformation**

Based on definition of complex-time vector by Park [2] the real- and imaginary parts of the vector can be obtained:

\[
u(t) = \frac{2}{3} \left[ u_a(t) + a u_b(t) + a^2 u_c(t) \right] = a e(t) + j b \beta(t),
\]

(18)

the real- and imaginary parts of the vector can be obtained:

\[u_a(t) = \frac{1}{3} \left[ 2u_a(t) - u_b(t) - u_c(t) \right],\]

(19)

\[u_b(t) = \frac{\sqrt{3}}{3} \left[ u_b(t) - u_c(t) \right].\]

(20)

Considering sum of phase voltages to be zero:

\[u_a(t) + u_b(t) + u_c(t) = 0,\]

(21)

real- and imaginary parts will be

\[u_a(t) = u_a(t), \quad u_b(t) = \frac{u_b(t) - u_c(t)}{\sqrt{3}},\]

(22)

under considering sum of phase voltages to be zero.

Voltage of \( a \)-phase of the inverter then will be

\[u_a(t) = u_a(t) = \frac{4U_0}{\pi} \sum_{\nu=0}^{\infty} \frac{\sin \left(2\nu+1\right) \frac{2\pi}{3} \cos \left(2\nu+1 \left(\omega t - \frac{\pi}{3}\right)\right)}{2\nu+1},\]

(23)

Voltage of \( b \)-phase lags the voltage of \( a \)-phase, thus
\[
\sqrt{3} u_\beta(t) = u_b(t) - u_c(t) = \frac{-8U_0}{\pi} \sum_{v=0}^{\infty} \sin^2 \left( \frac{(2v+1)\pi}{3} \right) \sin \left[ \frac{(2v+1)(\omega t + \frac{2\pi}{3})}{2v + 1} \right] \tag{24}
\]

**Note:** it is to be aware of the fact that Eqs. (23) and (24) create orthogonal series for \( u_\phi \) and \( u_\beta \) [3], [6-8], which can be processed by orthogonal Fourier series rules.

**Determination of phase current of load in steady-state**

Based on (19), (20) the differences of the phase voltages are \( u_\alpha - u_c \), indeed, the \( \beta \)-components of Park complex time-vector, multiplied by constant \( \sqrt{3} \) by (24) and the difference of phase-voltages \( u_a(t) - u_b(t) \) then similarly will be

\[
u_a(t) - u_b(t) = \frac{-8U_0}{\pi} \sum_{v=0}^{\infty} \sin^2 \left( \frac{(2v+1)\pi}{3} \right) \sin \left[ \frac{(2v+1)(\omega t + \frac{2\pi}{3})}{2v + 1} \right] \tag{25}
\]

Since \( u_a(t) - u_b(t) = u_i + (R + j(2v+1)\alpha L) \cdot [i_{ab}(t) - i_{ca}(t)] - (i_{bc}(t) - i_{ab}(t))] \) and \( i_{ab}(t) + i_{bc}(t) + i_{ca}(t) = 0 \),

\[
u_a(t) - u_b(t) = u_i + 3 \cdot i_{ab}(t) \cdot (R + j(2v+1)\alpha L). \tag{26}
\]

Load current \( i_{ab} \) then will be

\[
i_{ab}(t) = \frac{u_a(t) - u_b(t) - u_i}{3 \cdot (R + j(2v+1)\alpha L)} = \frac{u_a(t) - u_b(t) - u_i}{3 \cdot Z_{2v+1} e^{j(2\pi v+\alpha)\omega t}}. \tag{28}
\]

Considering rotor speed equal zero and substituting time-derivative operator \( \alpha \) p by \( j \cdot \omega \) voltages \( u_{d,q} = u_{ab} \cdot e^{\alpha} \).

**Model of induction machine**

The key assumptions in the model of induction motor are:
- stator windings are distributed to produce sinusoidal MMF in space,
- the rotor bars or coils are arranged such that at any instant, the rotor MMF waves are sinusoidal in space and have the same number of poles as the corresponding stator MMF wave,
- the air gap is uniform,
- the magnetic circuit is linear.

The equations of the machine are:

\[
u_{\alpha s} = pA_{\alpha s} + r_1 \cdot i_{\alpha s}, \tag{31}
\]

\[
u_{\beta s} = pA_{\beta s} + r_1 \cdot i_{\beta s}, \tag{32}
\]

\[
0 = pA_{\alpha r} + r_2 \cdot i_{\alpha r} + \omega_0 \cdot A_{\alpha r}, \tag{33}
\]

\[
0 = pA_{\beta r} + r_2 \cdot i_{\beta r} - \omega_0 \cdot A_{\alpha r}. \tag{34}
\]

In the equations \( \omega_0 \) -- the rotor speed, \( p \) -- the time-derivative operator, \( A \) --the fluxes linking the subscripted windings, \( A_{\alpha r} \) -- rotating voltage.

Voltages of stator and rotor can be derived for \( d,q \) voltages

\[
u_{d,q} = u_{ab} \cdot e^{\alpha}. \tag{35}
\]

**Simulation experiments results**

Based on above mentioned equation (23)-(35) the following simulations in Gauss plane and in time-domain have been programmed in Matlab programming environment.

Parameters of the circuit: \( R = 1 \) Ohm, \( L = 5 \) mH, \( U_0 = 100 \) V, \( \omega_0 \) = f (\( \omega_0 \)).

Parameters of the simulation: time increment \( \Delta T = 1 \) \( \mu s \), number of considered odd harmonic components from 1 up to 999. Version of Mat Lab programming environment: R2007b.
Conclusions

The relation for resulting current wave-form can be obtained also in compact closed form using classical analytical solution, Laplace transform and similar methods.

Anyway, the solution given in the paper makes possible to analyse more exactly effect of each harmonic component comprised in total waveform on induction motor quantities.

The simulation experiments have shown very good coincidences of theoretical and simulated results.

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References


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